MTH 303
Real analysis **Homework 2**

1. Let $a \in \mathbb{R}$. If x belongs to the neighborhood $N_\epsilon(a)$ for every $\epsilon > 0$, then show that $x = a$.

2. If $E \subseteq \mathbb{R}$ be a nonempty bounded set, and $I_E = [\inf E, \sup E]$, show that $E \subseteq I_E$. Moreover, if I is any other closed bounded interval containing E, then $I_E \subseteq I$.

3. Construct a bounded set of real numbers with exactly five limit points.

4. A set E is closed if and only if its complement is open.

5. Let $\{E_\alpha\}_{\alpha\in\Lambda}$ be a (finite or infinite) collection of sets E_α. Then
$$(\cup_\alpha E_\alpha)^c = \cap_\alpha E_\alpha^c.$$

6. (a) For any collection $\{O_\alpha\}_{\alpha\in\Lambda}$ of open sets , $\cup_\alpha O_\alpha$ is open.
   (b) For any collection $\{F_\alpha\}_{\alpha\in\Lambda}$ of closed sets , $\cap_\alpha F_\alpha$ is closed.
   (c) For any finite collection $\{O_k\}_{k=1}^n$ of open sets , $\cap_{k=1}^n O_k$ is open.
   (d) For any finite collection $\{F_k\}_{k=1}^n$ of closed sets , $\cup_{k=1}^n F_k$ is closed.

7. Let E' denote the set of all limit points of the set E. Show that
   (a) the set E' is closed.
   (b) E and $\bar{E}$ have the same limit points.
   (c) Do E and E' always have the same limit points? Justify your answer.

8. Let $\{A_k\}_{k\in\mathbb{N}}$ be a sequence of subsets in $\mathbb{R}$.
   (a) If $B_n = \cup_{k=1}^n A_k$, show that $\bar{B}_n = \cup_{k=1}^n \bar{A}_k$, for all $n \in \mathbb{N}$.
   (b) If $B = \cup_{k\in\mathbb{N}} A_k$, show that $\cup_{k\in\mathbb{N}} \bar{A}_k \subseteq \bar{B}$.
   Moreover, show that this inclusion can be proper.

9. Is every point of every open set E in $\mathbb{R}$ a limit point of E? Justify your answer.

10. Let E be a nonempty bounded set of real numbers. Show that $\inf E$ and $\sup E$ belong to $\bar{E}$.

11. Let E^0 denote the interior (set of all interior points) of a set E.
   (a) Prove that E^0 is always open.
   (b) Prove that E is open if and only if $E^0 = E$.
   (c) Prove that E^0 is the largest open set contained in E.
   (d) Prove that the complement of E^0 is the closure of the complement of E, i.e., $(E^0)^c = \overline{E^c}$.
   (e) Do E and $\bar{E}$ always have the same interiors?

(f) Do E and E^0 always have the same closures?

12. Prove that every open set in $\mathbb{R}$ can be written as the disjoint union of at most countable collection of open intervals.

13. Closed subsets of compact sets are compact.

14. If F is closed and K is compact, then $F \cap K$ is compact.

15. Let $K = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$. Prove that K is compact (without using the Heine-Borel theorem).

16. Construct a compact set of real numbers whose limit points form a countable set.

17. Give three different examples of open covers of the interval $(0, 1)$ which do not admit finite sub-covers.

18. If $\{K_n\}_{n\in\mathbb{N}}$ be a sequence of nonempty compact sets such that $K_{n+1} \subseteq K_n$, $\forall\ n \in N$, then show that $\cap_{n\in\mathbb{N}} K_n \neq \emptyset$.

19. Prove that
    (a) The total length of the removed intervals in the construction of the Cantor set is 1.
    (b) The Cantor set P does not contain any nonempty interval.
    (c) The Cantor set is perfect.
    (d) The Cantor set is uncountable.

20. Define a point $x \in \mathbb{R}$ to be a condensation point of a set $E \subseteq \mathbb{R}$ if every neighborhood of x contains uncountably many points of E.

    Suppose $E \subseteq \mathbb{R}$ be an uncountable set, and let P be the set of all condensation points of E. Prove that P is perfect and that at most countably many points of E are not in P.

21. Prove that every closed set in $\mathbb{R}$ is the union of a (possibly empty) perfect set and a set which is at most countable.

22. Every countable closed set in $\mathbb{R}$ has isolated points.
    Isolated point: If $x \in E \subseteq \mathbb{R}$ and x is not a limit point of E, then x is called an isolated point of E.

23. Is there a nonempty perfect set in $\mathbb{R}$, which contains no rational number?